Definitions and key facts for section 1.7

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. Otherwise, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent**, that is, if there exist weights $c_1, c_2, \ldots, c_p$ not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

In this case, we call (1) a **linear dependence relation**.

Fact: Linear independence of the columns of a matrix
The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution which, in turn, occurs if there are no free variables.

Fact: Characterization of linearly dependent sets A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

Indeed, if $\mathbf{v}_1 \neq \mathbf{0}$ and S is linearly dependent then some $\mathbf{v}_j$ is a linear combination of the *preceding* vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.
So, a pair of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if one is a multiple of the other, and linearly independent if neither is a multiple of the other.

Fact: Two conditions guaranteeing linear dependence Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$, then

- if $\mathbf{0}$ is in S, S is linearly dependent; and
- if $p > n$, S is linearly dependent. (That is, if there are more vectors than the number of entries in each vector, they are linearly dependent.)